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Some analytical solutions for propagation of gravity waves along a small-scale frontal surface

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Propagation of the gravity waves along a small-scale frontal surface is considered. The simplified perturbation equations, which can be applied to describe this phenomenon, are introduced and a method to solve them is exposed.

The main difficulty in finding analytical solution to the resulting initial value problem is in taking into account the frontal inclination towards ground. It is shown how this problem can be overcome by an approximate solution in form of a Bessel-Fourier series, provided the inclination is partially ignored in the governing equations.

In order to demonstrate the role of the frontal inclination in the propagation of disturbances along the frontal surface, some properties of the exact solution, which can be obtained after substitution of the wavelike solution in the complete system, are analysed. It is shown that inclination leads to the appearance of an instability of downstream propagating waves. This phenomenon is further described and discussed.

Neka rešenja za prostiranje gravitacionih talasa duž frontalne površine malih razmera

Razmatrano je prostiranje gravitacionih talasa duž frontalne površine malih razmera. Postavljen je jednostavan sistem poremećajnih jednačina koje opisuju ovaj fenomen i izložen je metod za njihovo rešavanje.

Teškoće koje se ovde susreću, ukoliko se dati sistem želi rešiti kao problem početne vrednosti, povezane su sa uzimanjem u obzir nagiba frontalne površine prema tlu. Pokazano je kako se ovaj problem može prevladati postavljanjem približnog rešenja u obliku Bessel-Fourierovog reda, pri čemu se nagib delimično zanemaruje u polaznim jednačinama.

Sa ciljem da se detaljnije istraži uloga frontalnog nagiba na način prostiranja poremećaja, analizirane su neke osobine tačnog rešenja polaznog sistema. Tom prilikom pokazalo se da nagib zapravo dovodi do pojave nestabilnosti onih talasa koji se prostiru nizvodno. Ovaj fenomen je dalje opisan i prodiskutovan.

1. Introduction

An evidence of the existence of gravity waves propagating along the frontal surfaces are separate precipitation zones which can be observed behind some cold fronts (for example Peng-Yun Wang et al., 1983). Namely, these waves, which are often interpreted as a consequence of the vertical shearing instability, (Orlanski, 1960; Vitek, 1969), lead to the development of organized cloudiness along the frontal surface. This, in turn, leads to the appearance of mentioned separate precipitation zones.

In this paper, propagation of such gravity waves will be considered, using a simple shallow water equations model. Some conclusions which may be of interest will be shown.

2. Governing equations

For the sake of simplicity, Coriolis parameter will be omitted here. This means that our discussion will be restricted to the case of a meso-γ front, (Orlanski, 1975), or to the leading edge of a synoptic front. The equations describing homogeneous fluid, with no vertical shear, will be applied to the lower, cold air. Thus, we shall use the modified value of gravity

$$g^* = g \frac{\Delta \rho}{\rho} \tag{1}$$

Here, $\Delta\rho$ is the difference in density between the lower and the upper air, and other symbols, when not stated explicitly, have their usual meaning. Furthermore, we shall assume that the frontal surface, given in Fig. 1, moves with a constant velocity U. We shall take this to represent our basic state. It can be described by the

$$g^* \frac{\partial H}{\partial x} = -KU$$

$$\frac{\partial H}{\partial t} = -U \frac{\partial H}{\partial x} \tag{2}$$

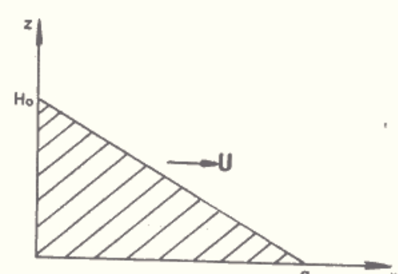

Figure 1. Considered meso-scale frontal surface and some of the notation used.

where K is the Rayleigh „friction" coefficient. Such an approximation to the friction term in the shallow water equations model was discussed by Rančić (1985). A solution for the basic state height, $H(x, t)$, satisfying this system, can be readily found in the form

$$H = H_O \left(1 - \frac{x - Ut}{a}\right), \quad \text{for } x \leqslant a + Ut;$$
$$H = 0, \qquad \text{for } x > a + Ut, \tag{3}$$

where both H_O and a depend on g^*, U and K.

The behaviour of small perturbations of such a basic state will be considered. They are described by the linearized equations in the form

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} = -g^* \frac{\partial h}{\partial x} - Ku$$
$$\frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} = -H \frac{\partial u}{\partial x} - \underline{u \frac{\partial H}{\partial x}} \tag{4}$$

where u and h denote perturbation variables. The last term in $(4)_2$ is underlined for later convenience. It should be noted that the above system is valid only at some distance away from the ground, where the assumption of the linearization procedure, $h \ll H$, is justified.

3. Solution of the perturbation system

a) Ignoring of the term $u\, \partial H / \partial x$

We shall first consider the case where the inclination of the frontal surface, $\partial H/\partial x$, is sufficiently small so that the last term in the continuity equation can be ignored in comparison with other terms in this equation. Note that, thereby, inclination of the frontal surface is not entirely ignored. Namely, dependance of the basic state height of the free surface. H. on the x coordinate, is still retained in the remaining term on the right hand side of the continuity equation.

In dealing with (4), it is convenient to introduce the system of coordinates which moves with the basic state velocity U. Thus, we define

$$x' = x - Ut, \quad u' = u - U.$$
$$t' = t. \qquad h' = h.$$

Furthermore, nondimensional variables

$$x'' = \frac{x'}{a}, \quad t'' = \frac{t'}{T} \quad h'' = \frac{h'}{H_0}$$

may be introduced, where

$$T = \frac{a}{(g^* H_0)^{1/2}}$$

Defining, in addition $2p \equiv KT$, and omitting the double prime signs, we arrive at

$$\frac{\partial^2 h}{\partial t^2} + 2p\frac{\partial h}{\partial t} = (1-x)\frac{\partial^2 h}{\partial x^2} \tag{5}$$

With the help of

$$h(x,t) = X(x)\,Y(t),$$

this equation can be transformed into

$$\begin{aligned} &\frac{d^2 X}{dx^2} + \frac{\lambda^2{}_k}{(1-x)}X = 0 \\ &\frac{d^2 Y}{dt^2} + 2p\frac{dY}{dt} + \lambda^2{}_k Y = 0, \end{aligned} \tag{6}$$

where $\lambda^2{}_k$ is a separation constant. We shall solve (6) under a somewhat restrictive boundary condition that the perturbation height is always equal to zero at the place where frontal surface touches the ground ($x = 1$) and at one more place along the front, say at $x = 0$. Thus, we require

$$h(0,t) = h(1,t) = 0 \tag{7}$$

To solve now the first of Eqs. (6), we shall make use of the substitution

$$\xi = (1-x)^{1/2}$$

so that $(6)_1$ takes the form

$$\frac{d^2 X}{d\xi^2} - \frac{1}{\xi}\frac{dX}{d\xi} + \lambda^2{}_k X = 0 \tag{8}$$

The solution of (8) is

$$X = \xi J_1\,(2\lambda_k \xi) \tag{9}$$

where J_1 is the Bessel function of the first kind (Mitrinović, 1975). We see that (9) satisfies boundary condition $(7)_2$. To satisfy the remaining boundary condition, $(7)_1$, the separation constant should be defined as

$$\lambda^2{}_k = z_k/2$$

where z_k is a positive zero of the Bessel function of the first kind.

In view of the ortogonality property of the Bessel functions, an arbitrary function $f(\xi)$, may be written as

$$f(\xi) = \sum_{k=1}^{\infty} A_k\,\xi J_1\,(z_k \xi),$$

where the coefficients A_k are defined as

$$A_k \equiv \frac{2}{J_2\,(z_k)^2}\int_0^1 f(\xi) J_1\,(z_k\xi)\,d\xi$$

Here, J_2 is the Bessel function of the second kind.

Now, the general solution of the (5) can be written in the form

$$h(x,t) = \sum_{k=1}^{\infty} A_k \exp(-pt) \cos \omega_k t \cdot (1-x)^{1/2} J_1 [z_k(1-x)^{1/2}], \qquad (10)$$

where

$$\omega_k = (z_k/4 - p^2)^{1/2}$$

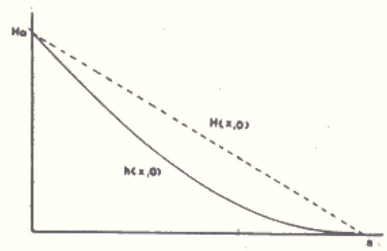

Figure 2. The initial height of the basic state, *H(x)*, and the perturbation, *h(x)*, used to calculate frontal shapes shown in Fig. 3.

For the numerical values H_o = 1 km, a = 5 km and g^* = 0.2 m/s^2, and the initial condition field as given in Fig. 2, an example of the solution (10) is calculated. Results obtained for several successive moments of time are shown in Fig. 3.

As can be seen, the perturbations propagate along the frontal surface as a *standing wave,* which is a consequence of the assumed boundary conditions. The oscillations of the frontal surface are *damped with time,* as a consequence of the friction term.

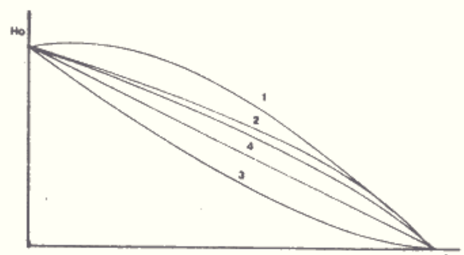

Figure 3. The frontal surface in case when the basic state inclination of the frontal surface is ignored. Curves marked by 1, 2, 3 and 4, represent shapes of the frontal surface after 5. 10. 15 and 25 minutes, respectively.

b) Analysis with the presence of the term u ∂H/∂x

It is, however, interesting to see what happens when the term $u\,\partial H/\partial x$ is not ignored in the governing equations; that is, what is the total effect of the inclination of the frontal surface on the propagation of gravity waves.

After substitution of the wavelike solutions, $e^{ik(x-ct)}$, in the complete system (4), the phase velocity of the gravity waves can be found in the form

$$c = c_r + ic_i, \tag{11}$$

where

$$c_r = U + \alpha$$
$$c_i = -K/2k + \beta, \tag{12}$$

and

$$2\alpha^2 = [(g^*H - K^2/4k^2)^2 + (KU/k)^2]^{1/2} + (g^*H - K^2/4k^2) \tag{13}$$
$$2\beta^2 = [(g^*H - K^2/4k^2)^2 + (KU/k)^2]^{1/2} - (g^*H - K^2/4k^2)$$

Thus, the waves are seen to propagate upstream and downstream relative to the basic wind. Moreover, the downstream propagating waves, (i. e., those which are connected with the „+" sign of the solutions for α and β from the Eqs. (13)), *become unstable,* if the condition

$$\beta > K/2k \tag{14}$$

is fulfilled. Inserting β from (13) into (14), it takes the form

$$F > 1 \tag{15}$$

where F is the Froude number ($F \equiv U^2/g^*H$).

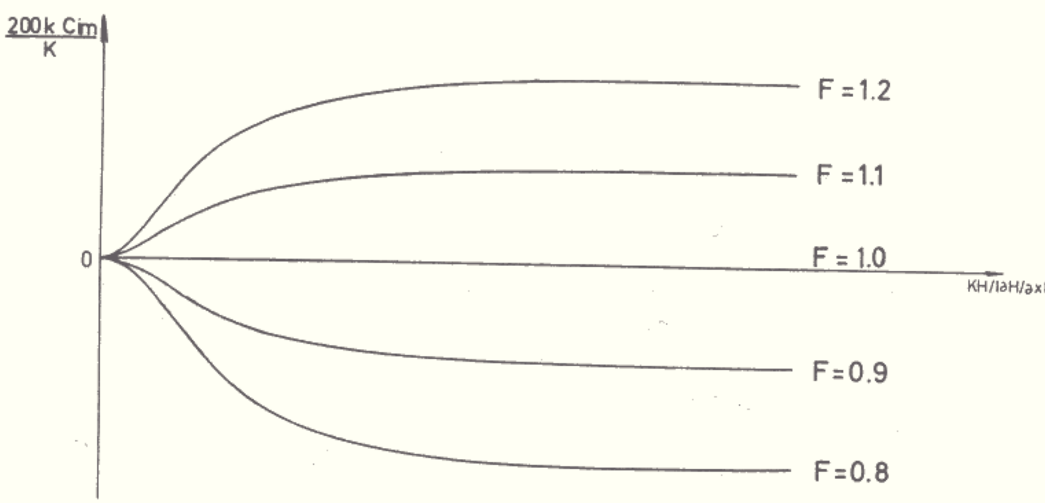

Figure 4. Nondimensional growth-rate, 200 kc$_i$/K, as a function of nondimensional wave number, $kH/|\partial H/\partial x|$, for several values of the Froude number.

Thus, *the downstream propagating waves always become unstable* as they approach the ground, and attain a sufficiently small value of H.

Note that if this analysis were to be performed after ignoring the term $u\,\partial H/\partial x$, β would be reduced to zero, and no instability would exist. Thus, *instability is just a consequence of the frontal inclination, i. e., of the underlined term in Eqs. (4),* ignored in the preceding subsection.

Nondimensional growth-rate, $\delta \equiv 2kc_i/K$, is given by the expression

$$2(\delta + 1)^2 = [(4FX^2 - 1)^2 + 16F^2X^2]^{1/2} - (4FX^2 - 1), \qquad (16)$$

where $X \equiv kH/|\partial H/\partial x|$. This is shown in Fig. 4, for several values of the Froude number. Note that $\delta \to 1$ when $X \to \infty$. Thus, as can be seen, the shorter are the waves, the more is increase (or decrease, depending on the value of the Froude number).

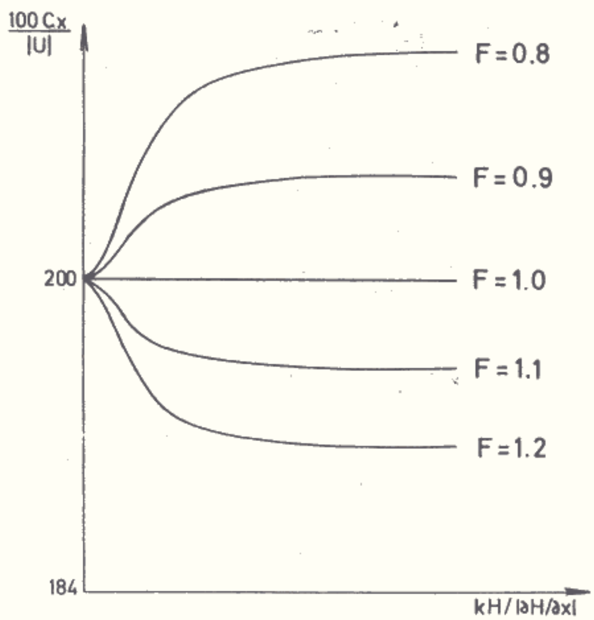

Figure 5. Nondimensional propagation speed of unstable perturbations, c_r/U, as a function of nondimensional wavenumber, $kH/|\partial H/\partial x|$, for several values of the Froude number.

Propagation speed of the unstable perturbations, c_r/U, is given by the expression

$$(c_r/U - 1)^2 = \frac{[(4FX^2 - 1) + 16F^2X^2]^{1/2} + (4FX^2 - 1)}{8\,F^2X^2}. \qquad (17)$$

and it is shown in Fig. 5. Note that $c_r/U \to 1/\sqrt{F} + 1$ when $X \to \infty$. Thus, the downward propagating waves deccelerate with the increase of the amplitudes.

4. Concluding remarks

As has been shown, if the presumably small term $u\,\partial H/\partial x$ is ignored in the system which describes propagation of gravity waves along the frontal surface, then an important property of downward propagating waves is lost. Therefore, instability here is not a consequence of the shear, but rather of the inclination of the frontal surface. Note that the basic state height is in a dynamical balance. This balance is made by the equilibrium of the pressure gradient force and the friction term. At the same time, basic state is not energetically stable, i. e. it does not have the minimum of the potential energy. Consequently, the perturbations which propagate along the frontal surface, tend to lead the basic state to this minimum. The amplitudes of downstream (relative to the basic state wind) propagating waves, increase and their phase speed deccelerates; the amplitudes of upstream propagating waves, however, decrease and their phase speed accelerates.

In other words, the basic state has some available potential energy, which amplitudes of downward propagating waves use for their growth.

To illustrate this energy transformation mechanism, we can consider a simple model, similar to that used by Ćurić (1980), and shown in Fig. 6. Potential energy of the closed system shown in the figure is given by

$$E_p = Mgz_c \tag{18}$$

where z_c is the height of the center of the entire mass of the system, denoted by M. In a nondimensional form, it can be expressed as

$$\frac{2\,E_p}{\mathrm{b\,g}\,\rho_2} = \frac{(\Delta - 1)}{12}\;(3 + 4x^2) + 1 \tag{19}$$

where $x = \eta/b$, η and b being defined in the figure, and $\Delta \equiv \rho_1/\rho_2$. Note that x takes values in the interval $(-1/2, 1/2)$.

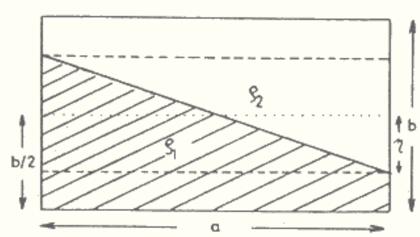

Figure 6. A simple model which represents closed system of two fluids with the different densities, and the inclined boundary surface between them. The system has an availabe potential energy which perturbations use for their growth.

The available potential energy as a function of nondimensional parameter X is shown in Fig. 7. As can be seen, it has a minimum for $X = 0$, i. e., for the horizontal position of the boundary surface.

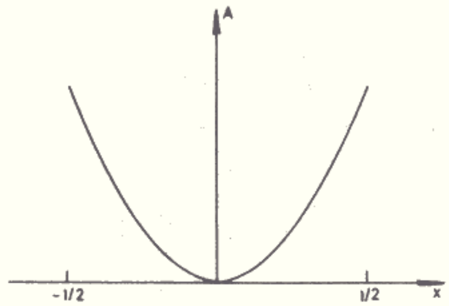

Figure 7. Available potential energy of the system shown in Fig. 6, as a function of nondimensional parameter η/b.

Thus, the gravity waves which propagate down the front, seem to tend to reduce the slope of the boundary surface, forcing it to take the horizontal position, characterized by the minimum of potential energy.

In that sense, the considered mechanism is the same as that of baroclinic instability, well known and first described by Lorenz (1955). Typically, the baroclinic instability mechanism is used in the context of the quasi-geostrophic equations to describe development of the large scale systems. Here, it has been applied to the case of a small scale perturbations, using a rather simple physical model.

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References

Ćurić, M. (1980): Dynamics of a cold air outflow from the base of thunderstorm. A simple model. Journal de Recherches Atmospheriques **14**, 493–498.

Lorenz, E. N. (1955): Available potential energy and the maintance of the general circulation. Tellus **7**, 157–167.

Mitrinović, D. S. (1975): Uvod u specijalne funkcije. Građevinska knjiga, Beograd, 1975, 231 pp, (in Serbo-Croat.).

Orlanski, I. (1975): A rational subdivision of scales for atmospheric processes. Bulletin of the American Meteorological Society **56**, 527–530.

Peng-Yun Wang, D. P. Parsons and P. V. Hobbs (1983): The mesoscale and microscale structure and organisation of clouds and precipitation in middlatitude cyclons. VI. Wavelike rainbands with a cold-frontal zone. Journal of the Atmospheric Sciences **40**, 543–558.

Rančić, M. (1985): Neka rešenja za oblik granične površine nastale razlivanjem hladnog vazduha ispod baze kumulonimbusa. Institut za Meteorologiju, Univerzitet u Beogradu, 1985, 155 pp. (Magistarski rad, in Serbo-Croat.).

Vitek, V. (1969): A simple model of certain type of frontal wave in the atmosphere. Tellus **21**, 724–735.

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